Exact solutions of the Lawrence-Doniach model for layered superconductors

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We solve the problem of exact minimization of the Lawrence-Doniach (LD) free-energy functional in parallel magnetic fields. We consider both the infinite in the layering direction case (the infinite LD model) and the finite one (the finite LD model). We prove that, contrary to a prevailing view, the infinite LD model does not admit solutions in the form of isolated Josephson vortices. For the infinite LD model, we derive a closed, self-consistent system of mean-field equations involving only two variables. Exact solutions to these equations prove simultaneous penetration of Josephson vortices into all the barriers, accompanied by oscillations and jumps of the magnetization, and yield a completely new expression for the lower critical field. Moreover, the obtained equations allow us to make self-consistent refinements on such well-known results as the Meissner state, Fraunhofer oscillations of the critical Josephson current, the upper critical field, and the vortex solution of Theodorakis [S. Theodorakis, Phys. Rev. B **42**, 10172 (1990)]. Our consideration of the finite LD model illuminates the role of the boundary effect. In contrast to the infinite case, an explicit analytical solution to the Maxwell equations of the finite case does not preclude the existence of localized Josephson vortex configurations. By the use of this solution, we obtain a self-consistent description of the Meissner state. Finally, we discuss some theoretical and experimental implications.

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I. INTRODUCTION

We obtain exact analytical solutions to the phenomenological Lawrence-Doniach [1] (LD) model for layered superconductors in external parallel magnetic fields. We consider both the infinite in the layering direction case (the infinite LD model) and the finite one (the finite LD model). This paper should be considered as a logical continuation of our previous study of layered superconductors on the basis of a microscopic approach [2].

At present, the LD model is widely used for the description of low-T_c layered superconductors and superlattices as well as high-T_c superconductors exhibiting the intrinsic Josephson effect [3,4]. Surprisingly, despite a large number of theoretical publications on this subject, it has not been realized yet that the problem of the parallel magnetic field is exactly solvable. Up to now, actual analytical solutions with different degrees of accuracy have been obtained only for relatively simple particular cases of the infinite LD model: the Meissner state [5], Fraunhofer oscillations of the critical Josephson current [6], the upper critical field $H_{c2}(T)$ [7,3], and the vortex state [8] in the intermediate field regime.

Unfortunately, the calculations of the lower critical field H_{c1} [9,10], based on the assumption of isolated Josephson vortex penetration, raise questions. In these calculations, one employs an anisotropic continuum approximation outside the so-called "Josephson vortex core region" [11], completely neglecting the intrinsic discreteness of the LD model. As has been recently shown by Farid [12], a set of equations thus obtained has no physical solution. Furthermore, the calculations of a triangular Josephson vortex lattice [13], also based on the assumption of the existence of isolated Josephson vortices, are at odds with the exact vortex solution of Theodorakis [8], valid in the same field range and exhibiting full homogeneity in the layering direction.

As has been shown within the framework of the microscopic theory [2], the resolution of these contradictions lies in the analysis of singular mathematical structure of free-energy functionals of layered superconductors. In particular, the system of the Maxwell equations in layered superconductors contains a constraint relation that physically constitutes the conservation law for the total intralayer current. According to this constraint relation, the phases of the superconducting order parameter (the pair potential) at different layers turn out to be mutually dependent. The minimization of the free energy with respect to the phases must necessarily take into account this fact. The neglect of mutual dependence of the phases leads to an incomplete set of mean-field equations. In the present paper, we elucidate this mathematical issue in full detail.

Section II of the paper is devoted to the infinite in the layering direction LD model. In subsection II.A, we concentrate on exact minimization of the LD free-energy functional. Using general field-theoretical arguments, we

prove that the Maxwell equations of the LD model contain an infinite number of unphysical degrees of freedom that cannot be eliminated by imposing a gauge condition. We achieve the elimination of these redundant degrees of freedom by minimizing the free energy with respect to the phases, taking account of the conservation law for the total intralayer current. In this way, we obtain a complete set of necessary and sufficient conditions of an unconditional minimum of the LD functional. These conditions constitute a remarkably simple, closed, self-consistent system of mean-field equations involving only two variables: the reduced modulus of the pair potential (the same for all the superconducting layers) and the phase difference (the same for all the barriers). In addition, we prove that inhomogeneous in the layering direction field configurations do not correspond to any stationary points of the free energy. As a result, contrary to the prevailing view [9,11,10,13], the infinite LD model does not admit any solutions in the form of isolated Josephson vortices.

In subsection II.B, we proceed to exact solutions of the mean-field equations of the infinite LD model describing major physical effects. We arrive at a new scenario of the flux penetration at H_{c1}: We show that Josephson vortices penetrate all the barriers simultaneously and coherently, forming homogeneous field distribution in the layering direction (a "vortex plane"). The corresponding lower critical field is $H_{c1} = 2\left(\pi e p \lambda_j\right)^{-1}$, where p is the layering period, $\lambda_J = \left(8\pi e j_0 p\right)^{-1/2}$ is the Josephson penetration depth, with j_0 being the critical density of the Josephson current. We show that the magnetization exhibits oscillations and jumps due to successive vortex plane penetration. We also obtain all well-known limiting cases [the Meissner state, Fraunhofer oscillations of the critical Josephson current, the upper critical field $H_{c2}(T)$, and the vortex solution of Theodorakis] with self-consistent refinements. All these results stand in complete agreement with our previous microscopic consideration [2].

In section III, we consider the finite, both in the layering direction and along the layers, LD model. We show that the emergence of additional boundary conditions in this case completely eliminates unphysical degrees of freedom of the Maxwell equations and makes minimization with respect to the phases impossible. An explicit solution to the Maxwell equations obtained in this section, in contrast to the infinite case, does not preclude the existence of localized Josephson vortex configurations. As regards the physical effects, we derive exact analytical expressions for the order parameter, the currents and the local magnetic field describing the Meissner state.

In section IV, we present a brief summary of the obtained results and discuss some theoretical and experimental implications. In appendix A, we obtain an explicit solution to the Maxwell equations in the infinite case. We also consider a variation of this solution induced by variations of the phases. In Appendix B, we discuss relationship to the microscopic theory [2]. This discussion casts light on the actual domain of validity of the LD model.

II. THE INFINITE LD MODEL

In this section, we consider an infinite in the layering direction LD model. One of the dimensions of the system along the layers is taken to be finite, although it can be made arbitrarily large.

We begin by reminding basic features of the LD model [1,7]. In this model, the temperature T is assumed to be close to the "intrinsic" critical temperature T_{c0} of individual layers:

$$\tau \equiv \frac{T_{c0} - T}{T_{c0}} \ll 1. \tag{2.1}$$

The superconducting (S) layers are assumed to have negligible thickness compared to the "intrinsic" coherence length $\zeta(T) \propto \tau^{-1/2}$, the penetration depth $\lambda(T) \propto \tau^{-1/2}$, and the layering period p. Taking the layering axis to be x, choosing the direction of the external magnetic field $\mathbf{H}$ to be z [$\mathbf{H} = (0, 0, H)$], assuming homogeneity along this axis and setting $\hbar = c = 1$, we can write the LD free-energy functional as

$$\Omega_{LD}\left[f_n, \phi_n, \frac{d\phi_n}{dy}, A_x, A_y; H\right] = \frac{p H_c^2(T)}{4\pi} W_z \int_{L_{y1}}^{L_{y1}} dy \sum_{n=-\infty}^{+\infty} \left[-f_n^2(y) + \frac{1}{2} f_n^4(y)\right.$$

$$+\zeta^2(T)\left[\frac{df_n(y)}{dy}\right]^2 + \zeta^2(T)\left[\frac{d\phi_n(y)}{dy} - 2eA_y(np, y)\right]^2 f_n^2(y)$$

$$\left.+\frac{r(T)}{2}\left[f_{n-1}^2(y) + f_n^2(y) - 2 f_n(y) f_{n-1}(y) \cos\Phi_{n,n-1}(y)\right]\right]$$

$$+\frac{4e^2\zeta^2(T)\lambda^2(T)}{p}\int\limits_{(n-1)p}^{np} dx\left[\frac{\partial A_y(x,y)}{\partial x}-\frac{\partial A_x(x,y)}{\partial y}-H\right]^2\Bigg], \tag{2.2}$$

$$\Phi_{n,n-1}(y)=\phi_{n,n-1}(y)-2e\int\limits_{(n-1)p}^{np} dx A_x(x,y),$$

$$\phi_{n,n-1}(y)=\phi_n(y)-\phi_{n-1}(y).$$

Here $\mathbf{A}=(A_x,A_y,0)$ is the vector potential, continuous at the S-layers: $\mathbf{A}(np-0,y)=\mathbf{A}(np+0,y)=\mathbf{A}(np,y)$; W_z is the length of the system in the z direction ($W_z\to\infty$); $f_n(y)$ $[0\le f_n(y)\le 1]$ and $\phi_n(y)$ are, respectively, the reduced modulus and the phase of the pair potential $\Delta_n(y)$ in the nth superconducting layer:

$$\Delta_n(y)=\Delta(T)f_n(y)\exp\phi_n(y), \tag{2.3}$$

with $\Delta(T)$ being the "intrinsic" gap $[\Delta(T)\propto\tau^{1/2}]$; $H_c(T)$ is the thermodynamic critical field; $r(T)=2\alpha_{ph}\tau^{-1}$ is a dimensionless phenomenological parameter of the Josephson interlayer coupling ($0<\alpha_{ph}\ll 1$). The local magnetic field $\mathbf{h}=(0,0,h)$ obeys the relation

$$h(x,y)=\frac{\partial A_y(x,y)}{\partial x}-\frac{\partial A_x(x,y)}{\partial y}. \tag{2.4}$$

A. Exact minimization of the LD functional

Our task now is to establish a closed, complete, self-consistent system of mean-field equations for the pair potential Δ_n and the local magnetic field h, which is mathematically equivalent to the minimization of (2.2) with respect to f_n, ϕ_n, and $\mathbf{A}$. This problem should be approached with a great deal of caution because of singular mathematical structure of the functional (2.2), resulting from gauge invariance combined with discreteness. Thus, one must take account of the fact that variations with respect to ϕ_n and $\mathbf{A}$ are not independent. Moreover, variations with respect to ϕ_n at different layers in themselves turn out to be mutually dependent. Unfortunately, these crucial points have not been realized in previous literature. To clarify them, we consider partial variational derivatives with respect to ϕ_n, and A_x, A_y, formally obtained under the assumption of the independence of these variables.

As the functional (2.2) is invariant under the gauge transformation

$$\phi_n(y)\to\phi_n(y)+2e\eta(np,y),\;\; A_i(x,y)\to A_i(x,y)+\partial_i\eta(x,y),\;\; i=x,y,$$

where $\eta(x,y)$ is an arbitrary smooth function of x, y in the whole region $(-\infty<x<+\infty)\times[L_{y1}\le y\le L_{y2}]$, partial variational derivatives with respect to ϕ_n, and A_x, A_y are related by the fundamental identities

$$2e\frac{\delta\Omega_{LD}}{\delta\phi_n(y)}\equiv\frac{\partial}{\partial y}\frac{\delta\Omega_{LD}}{\delta A_y(np,y)}+\frac{\delta\Omega_{LD}}{\delta A_x(np+0,y)}-\frac{\delta\Omega_{LD}}{\delta A_x(np-0,y)}. \tag{2.5}$$

Being a consequence of Noether's second theorem, such identities are typical of any gauge theory [14]. They imply that the number of independent Euler-Lagrange equations is less than the number of variables. In other words, the system of Euler-Lagrange equations contains unphysical degrees of freedom whose number is equal to the number of Noether's identities. Unusual is, however, an infinite number of identities (2.5). Indeed, in continuum gauge theories [such as, e.g., the Ginzburg-Landau (GL) theory of superconductivity] the number of Noether's identities is equal to the number of independent parameters of the relevant gauge group. [In the case of superconductivity, we are dealing with the electromagnetic one-parameter group $U(1)$.] Thus, by imposing gauge conditions in continuum gauge theories, one completely eliminates all unphysical degrees of freedom. By contrast, in the discrete LD theory a single available gauge condition cannot eliminate an infinite number of unphysical degrees of freedom resulting from (2.5). The resolution of the problem of the remaining "infinity minus one" unphysical degrees of freedom lies in implicit mutual dependence of the variations with respect to the phases ϕ_n at different S-layers. Below, we demonstrate this dependence explicitly. [See relation (2.12).]

To finish with the discussion of (2.5), we point out that these same identities hold also for the LD model with decoupled S-layers [when $r(T) \equiv 0$]. However, now the number of unphysical degrees of freedom is equal to the number of physically independent systems [one identity (2.5) per independent S-layer]. A single gauge condition completely eliminates the arbitrariness of the Euler-Lagrange equations in this case.

We start by minimizing with respect to $\mathbf{A}$. Varying (2.2) with respect to A_x, A_y in the regions $(n-1)p < x < np$ under the condition $\delta A_x(x, L_{y1}) = \delta A_x(x, L_{y2}) = 0$ yields the Maxwell equations

$$\frac{\partial h(x,y)}{\partial y} = 4\pi j_{n,n-1}(y) \equiv 4\pi j_0 f_n(y) f_{n-1}(y) \sin \Phi_{n,n-1}(y), \tag{2.6}$$

$$\frac{\partial h(x,y)}{\partial x} = 0, \tag{2.7}$$

where $j_{n,n-1}(y)$ is the density of the Josephson current between the $(n-1)$th and the nth layers, $j_0 = r(T)p/16\pi e\zeta^2(T)\lambda^2(T)$. Minimization with respect to $A_y(np, y)$ leads to boundary conditions at the S-layers

$$h(np-0,y) - h(np+0,y) = \frac{p f_n^2(y)}{2e\lambda^2(T)} \left[\frac{d\phi_n(y)}{dy} - 2eA_y(np,y) \right]. \tag{2.8}$$

Equations (2.6)-(2.8) should be complemented by boundary conditions at the outer interfaces $y = L_{y1}, L_{y2}$. As we do not consider here externally applied currents in the y direction, the first set of boundary conditions follows from the requirement that the intralayer currents vanish at $y = L_{y1}, L_{y2}$:

$$\left[\frac{d\phi_n(y)}{dy} - 2eA_y(np,y) \right]_{y=L_{y1},L_{y2}} = 0. \tag{2.9}$$

Applied to Eqs. (2.8), these boundary conditions show that the local magnetic field at the outer interfaces is independent of the coordinate x: $h(x, L_{y1}) = h(L_{y1})$, $h(x, L_{y2}) = h(L_{y2})$. Boundary conditions imposed on h should be compatible with Ampere's law $h(L_{y2}) - h(L_{y1}) = 4\pi I$ obtained by integration of Eqs. (2.6) over y, where

$$I \equiv \int_{L_{y1}}^{L_{y2}} dy j_{n+1,n}(y) = \int_{L_{y1}}^{L_{y2}} dy j_{n,n-1}(y) \tag{2.10}$$

is the total Josephson current.

Differentiating (2.8) with respect to y and employing (2.6), we arrive at the current-continuity laws for the S-layers:

$$\frac{\partial}{\partial y} \left[f_n^2(y) \left[\frac{d\phi_n(y)}{dy} - 2eA_y(np,y) \right] \right]$$

$$= \frac{r(T)}{2\zeta^2(T)} f_n(y) \left[f_{n-1}(y) \sin \Phi_{n,n-1}(y) - f_{n+1}(y) \sin \Phi_{n+1,n}(y) \right]. \tag{2.11}$$

These relations may be interpreted as "the Euler-Lagrange equations for the phases" in the sense that they can be formally obtained by taking partial variational derivatives with respect to ϕ_n under conditions (2.9). However, actual minimization of (2.2) with respect to the phases must take account of mutual dependence of $\delta\phi_n(y)$ at different layers, as shown in what follows. [The fact that relations (2.11) follow directly from the Maxwell equations (2.6), (2.8) is a consequence of (2.5). Surprisingly, this trivial functional dependence of the current-continuity laws for the S-layers on the Maxwell equations has not been pointed out in the previous literature. [15]]

Adding Eqs. (2.11), integrating and using boundary conditions (2.9), we get the conservation law for the total intralayer current:

$$\sum_{n=-\infty}^{+\infty} f_n^2(y) \left[\frac{d\phi_n(y)}{dy} - 2eA_y(np,y) \right] = 0. \tag{2.12}$$

This key relation of our consideration has mathematical form of a constraint [16] on the derivatives of the phases and the y components of the vector potential at different S-layers. Unfortunately, the existence of the constraint

relation (2.12) in the system of the Maxwell equations (2.6)-(2.8) has not been noticed in previous publications, hence difficulties in establishing a complete set of necessary and sufficient conditions of an unconditional minimum of (2.2). We want to emphasize that the fundamental constraint relation (2.12) and its corollaries below [relations (2.14), (2.15)] should not be confused with auxiliary constraint relations imposed on independent variables in the standard variational problem of a conditional minimum [16]. All constraints of the LD model appear as a result of singular structure of the functional (2.2) itself. (See Refs. [17,18] for a thorough discussion of singular field theories of this type.)

According to main principles of the calculus of variations [16], to minimize (2.2) with respect to ϕ_n, we must first eliminate the constraint (2.12). Assuming that $f_m(y) > 0$, where m is an arbitrary layer index, we rewrite (2.12) as

$$2eA_y(mp,y) = \frac{d\phi_m(y)}{dy} + \frac{1}{f_m^2(y)} \sum_{n\neq m} f_n^2(y) \left[\frac{d\phi_n(y)}{dy} - 2eA_y(np,y)\right]. \tag{2.13}$$

Equation (2.13) expresses $A_y(mp,y)$ as a function of all $\frac{d\phi_n(y)}{dy}$. It should be substituted into (2.2). Now all $\delta\phi_n(y)$ can be considered as independent. Carrying out the variation under the conditions (2.9), we obtain

$$f_{m-1}(y)\sin\Phi_{m,m-1}(y) - f_{m+1}(y)\sin\Phi_{m+1,m}(y) = 0, \tag{2.14}$$

$$\frac{\partial}{\partial y}\left[f_n^2(y)\left[\frac{d\phi_n(y)}{dy} - 2eA_y(np,y)\right]\right] - \frac{\partial}{\partial y}\left[f_n^2(y)\left[\frac{d\phi_m(y)}{dy} - 2eA_y(mp,y)\right]\right]$$

$$= \frac{r(T)}{2\zeta^2(T)} f_n(y)\left[f_{n-1}(y)\sin\Phi_{n,n-1}(y) - f_{n+1}(y)\sin\Phi_{n+1,n}(y)\right], \quad n\neq m.$$

Comparing these equations with (2.11) and integrating with boundary conditions (2.9) for $n = m$ yields

$$\frac{d\phi_m(y)}{dy} - 2eA_y(mp,y) = 0. \tag{2.15}$$

Since m is an arbitrary layer index, relations (2.14), (2.15) hold for all $n = m = 0, \pm1, \pm2, \ldots$ Note that only one of the two sets of relations (2.14), (2.15) is independent. For example, relations (2.14) can be obtained by inserting (2.15) into (2.11) and vice versa. In turn, the number of independent relations (2.15) is exactly equal to "infinity minus one", because they obey the constraint (2.12). As expected, the correct minimization of (2.2) with respect to the phases completely resolves the problem of unphysical degrees of freedom contained in Eqs. (2.6)-(2.8). Physically, relations (2.15), which appear already in the case of decoupled layers, minimize the kinetic energy of the intralayer currents and, by (2.8), assure the continuity of the local magnetic field at the S-layers. [According to (2.7), h does not depend on x in the barrier regions. Thus, $h(x,y) = h(y)$ in the whole region $(-\infty < x < +\infty)\times[L_{y1} \le y \le L_{y2}]$.] Relations (2.14) constitute stationarity conditions for the Josephson term in (2.2) and assure the continuity of the Josephson current at the S-layers as required by (2.10).

The above results, in fact, prove that inhomogeneous in the layering direction field configurations [i.e., those that do not satisfy (2.14), (2.15)] do not correspond to any stationary points of the functional (2.2). Consider the variation of the solution of (2.6)-(2.9) for A_y in the gauge $A_x = 0$ on an interval $(m-1)p < x \le mp$, induced by a variation of the phase at the nth layer. According to (A5), we have $\delta A_y(x,y) = \frac{1}{2e}\frac{f_n^2(y)}{f_m^2(y)}\frac{d\delta\phi_n(y)}{dy}$. Such a variation does not affect the energy of the magnetic field in (2.2). If $n = m$, the variation of the kinetic energy of the intralayer currents vanishes, but the first-order variation of the Josephson term is nonzero. If $n \neq m$, the variation of the Josephson term vanishes, but now the first-order variation of the kinetic energy of the intralayer currents is nonzero. These first-order variations of (2.2) vanish if and only if the conditions (2.14), (2.15) are fulfilled (i.e., for homogeneous field configurations). Unfortunately, this general mathematical consideration unambiguously precludes the existence of isolated Josephson vortices [9–11,13] in the infinite LD model. It also explains the results of Farid [12], who has pointed out inconsistencies in a mathematical description of such hypothetical entities.

It is instructive to look at the incompleteness of the system (2.6)-(2.8) from a slightly different mathematical point of view. In the gauge $A_x = 0$, this system reduces to an infinite set of integrodifferential equations (A3) for the phase differences $\phi_{n,n-1}$ (for fixed f_n). There are no theorems of existence and uniqueness of a solution to an infinite set of such equations. By contrast, for a finite set, describing a finite in the layering direction layered superconductor, the existence and uniqueness of a solution can be proved by standard methods of functional analysis. The description of a finite layered superconductor implies the specification of boundary conditions on $\mathbf{A}$ at the "top"

and "bottom" S-layers, whereas the infinite LD model considered here does not impose any boundary conditions on $\mathbf{A}$ at $x \to \pm\infty$. Thus, the arbitrariness contained in Eqs. (2.6)-(2.8) is an intrinsic mathematical property, necessary to satisfy additional boundary conditions in the case of the finite LD model. This issue is discussed in more detail in section III.

Minimization with respect to f_n is straightforward. Under the condition that $\delta f_n(L_{y1})$, $\delta f_n(L_{y2})$ are arbitrary, we get

$$f_n(y) - f_n^3(y) + \zeta^2(T)\frac{d^2 f_n(y)}{dy^2}$$

$$= \frac{r(T)}{2}\left[2f_n(y) - f_{n+1}(y)\cos\Phi_{n+1,n}(y) - f_{n-1}(y)\cos\Phi_{n,n-1}(y)\right]$$

$$+ \zeta^2(T)\left[\frac{d\phi_n(y)}{dy} - 2eA_y(np,y)\right]^2 f_n(y), \tag{2.16}$$

$$\frac{df_n}{dy}(L_{y1}) = \frac{df_n}{dy}(L_{y2}) = 0. \tag{2.17}$$

Equations (2.6)-(2.8), (2.16) and (2.14) [or, equivalently, (2.15)] (with $m \to n$), together with boundary conditions (2.9), (2.8) and boundary conditions for $h(y)$, form a closed, complete set of necessary and sufficient conditions of all the stationary points of the functional (2.2). For example, the well-known maximum $\Omega_{LD} = 0$ for $H = I = 0$ (the normal state) trivially satisfies these conditions with $f_n = 0$. The absolute minimum $\Omega_{LD} = -\frac{H_c^2(T)V}{8\pi}$ (V is the volume of the system) for $H = I = 0$ also satisfies these conditions with $\Phi_{n+1,n} = 0$ and $f_n = 1$. Complemented by the requirement that the Josephson term be a minimum, these conditions become necessary and sufficient conditions of all the minima of (2.2) for $H \neq 0$, $I \neq 0$, provided that $\Omega_{LD} < 0$. (For $L_{y2} - L_{y1} < +\infty$, the Josephson term is bounded and thus has both minimum and maximum values.)

Indeed, the Josephson term is minimized automatically. The kinetic energy of the intralayer currents is minimized by (2.15). The energy of the magnetic field [the last term in (2.2)] reaches its minimum value for given H and I too. This term is non-negative and necessarily has a minimum determined by the condition that its first-order variation vanish. (No other stationary points are available.) In the gauge $A_x = 0$, the first-order variation of the magnetic-field energy has the form

$$\delta\Omega_{LD}^{mf}\left[A_y; H\right] = \frac{e^2 H_c^2(T)\zeta^2(T)\lambda^2(T)W_z}{\pi}\int\limits_{L_{y1}}^{L_{y2}} dy \sum_{n=-\infty}^{+\infty}\left\{-\int\limits_{(n-1)p}^{np} dx \frac{\partial^2 A_y(x,y)}{\partial x^2}\delta A_y(x,y)\right.$$

$$\left. + \left[\frac{\partial A_y}{\partial x}(np-0,y) - \frac{\partial A_y}{\partial x}(np+0,y)\right]\delta A_y(np,y)\right\}. \tag{2.18}$$

The vanishing of the volume variation in (2.18) (the first term on the right-hand side) is assured by the Maxwell equations (2.7). The surface variation [the second term on the right-hand side of (2.18)] vanishes by virtue of (2.8) and (2.15). Consider now the condensation energy in (2.2) (the sum of the first three phase- and field-independent terms). This energy reaches its absolute minimum for $f_n = 1$, i.e. when the right-hand side of (2.17) is identically equal to zero. The Josephson term and the kinetic energy of the intralayer currents induce spatial dependence and a reduction of f_n, which increases the condensation energy. This influence is minimized under the considered conditions: the second term on the right-hand side of (2.17) vanishes according (2.15) and the first term is minimal when the Josephson energy is a minimum.

Thus, we have proved that the above obtained conditions minimize all the terms of the functional (2.2): the condensation energy, the Josephson energy, the kinetic energy of the intralayer currents and the magnetic-field energy. Any deviation from a solution satisfying these conditions increases all these terms. As a result, the overall LD free energy increases, as should be the case for an unconditional minimum [16].

Now we proceed to the simplification of Eqs. (2.6)-(2.8), (2.16) and (2.14) (with $m \to n$). As the local magnetic field h does not depend on x in the whole region $(-\infty < x < +\infty) \times [L_{y1} \leq y \leq L_{y2}]$, the quantities f_n, $\Phi_{n,n-1}$ cannot depend on the layer index:

$$f_n(y) = f_{n-1}(y) = f(y),\ \Phi_{n+1,n}(y) = \Phi_{n,n-1}(y) = \Phi(y). \tag{2.19}$$

The remaining unphysical degree of freedom of Eqs. (2.6)-(2.8), (2.14), related to the gauge invariance, is eliminated by fixing the gauge:

$$A_x(x,y) = 0,\ A_y(x,y) \equiv A(x,y). \tag{2.20}$$

[Note that $\partial A/\partial x$ and $\partial^2 A/\partial x \partial y$ are continuous at the S-layers by virtue of (2.8), (2.15), and (2.6), (2.14).] The second set of relations (2.19) now yields $\phi_n(y) = n\phi(y) + \eta(y)$, where $\phi(y)$ is the coherent phase difference (the same at all the barriers), and $\eta(y)$ is an arbitrary function of y that can be set equal to zero without any loss of generality.

From (2.7), employing the continuity conditions for A, $\partial A/\partial x$ and relations (2.15), we obtain

$$A(x,y) = \frac{1}{2ep}\frac{d\phi(y)}{dy}x. \tag{2.21}$$

Making use of these results, we reduce the functional (2.2) to

$$\Omega_{LD}\left[f,\phi;H\right] = \frac{H_c^2(T)}{4\pi}W_x W_z \int\limits_{L_{y1}}^{L_{y2}} dy \left[-f^2(y) + \frac{1}{2}f^4(y) + \zeta^2(T)\left[\frac{df(y)}{dy}\right]^2\right.$$

$$\left. + r(T)\left[1-\cos\phi(y)\right]f^2(y) + 4e^2\zeta^2(T)\lambda^2(T)\left[\frac{1}{2ep}\frac{d\phi(y)}{dy} - H\right]^2\right], \tag{2.22}$$

where $W_x = L_{x2} - L_{x1}$. The desired closed, self-consistent set of mean-field equations for the pair potential $\Delta_n(y)$ and the local magnetic field $h(y)$ takes the form

$$\Delta_n(y) = \Delta f(y)\exp\left[in\phi(y)\right], \tag{2.23}$$

$$f(y) + \zeta^2(T)\frac{d^2 f(y)}{dy^2} - f^3(y) - r(T)\left[1-\cos\phi(y)\right]f(y) = 0, \tag{2.24}$$

$$\frac{df}{dy}(L_{y1}) = \frac{df}{dy}(L_{y2}) = 0, \tag{2.25}$$

$$\frac{d^2\phi(y)}{dy^2} = \frac{f^2(y)}{\lambda_J^2}\sin\phi(y), \tag{2.26}$$

$$\lambda_J = \left(8\pi e j_0 p\right)^{-1/2}, \tag{2.27}$$

$$h(y) = \frac{1}{2ep}\frac{d\phi(y)}{dy}, \tag{2.28}$$

$$j(y) \equiv j_{n,n-1}(y) \equiv j_0 f^2(y)\sin\phi(y) = \frac{1}{4\pi}\frac{dh(y)}{dy}, \tag{2.29}$$

where $h(y)$ should satisfy appropriate boundary conditions at $y = L_{y1}, L_{y2}$ with $I \equiv \int\limits_{L_{y1}}^{L_{y2}} dy j(y)$ [see Eq. (2.10) above].

Remarkably, the coherent phase difference ϕ (the same for all the barriers) obeys only one nonlinear second-order differential equation (2.26) with only one length scale, the Josephson penetration depth λ_J [Eq. (2.27)], as in the case of the Ferrell-Prange equation for a single junction [19]. [Mathematically, equation (2.26) is a solvability condition for the Maxwell equations.] Due to the factor f^2, equation (2.26) is coupled to nonlinear second-order differential equation (2.24) describing the spatial dependence of the superconducting order parameter f (the same for all the S-layers). Equations (2.25) constitute boundary conditions for (2.24). The Maxwell equations (2.28), (2.29), combined together, yield Eq. (2.26), as they should by virtue of self-consistency.

It is important to note that equations (2.23)-(2.29), with an appropriate microscopic identification of $r(T)$ and j_0, can be considered as a limiting case of the true microscopic equations [2]. (See Appendix B for more details.)

Equations (2.24)-(2.29), together with (2.22), encompass the whole physics of the infinite LD model in parallel magnetic fields. They admit exact analytical solutions for all physical situations of interest. These solutions are discussed in the next subsection.

B. Major physical effects

1. The Meissner state

Consider a semi-infinite (in the y direction) LD superconductor with $r(T) \ll 1$, $L_{y1} = 0$, $L_{y2} \to +\infty$ in the external fields

$$0 \leq H \leq H_s = (ep\lambda_J)^{-1}, \tag{2.30}$$

In the Meissner state, $j(y) \to 0$, $h(y) \to 0$ for $y \to +\infty$. The requirement that the Josephson term in (2.22) be a minimum means that the density of the Josephson energy should vanish at $y \to +\infty$. This leads to the boundary conditions

$$\frac{d\phi}{dy}(0) = 2epH, \quad \frac{d\phi}{dy}(+\infty) = 0, \quad \phi(+\infty) = 0, \quad f(+\infty) = 1. \tag{2.31}$$

The solution of Eqs. (2.24), (2.26), (2.28), (2.29), subject to (2.25) and (2.31), up to first order in the small parameter $r(T)$ has the form

$$\phi(y) = -4\arctan\frac{H\exp\left[-\frac{y}{\lambda_J}\right]}{H_s + \sqrt{H_s^2 - H^2}}, \tag{2.32}$$

$$h(y) = \frac{2HH_s\left[H_s + \sqrt{H_s^2 - H^2}\right]\exp\left[-\frac{y}{\lambda_J}\right]}{\left[H_s + \sqrt{H_s^2 - H^2}\right]^2 + H^2\exp\left[-\frac{2y}{\lambda_J}\right]}, \tag{2.33}$$

$$j(y) = -\frac{HH_s}{2\pi\lambda_J}\left[H_s + \sqrt{H_s^2 - H^2}\right]$$

$$\times\frac{\left[\left[H_s + \sqrt{H_s^2 - H^2}\right]^2 - H^2\exp\left[-\frac{2y}{\lambda_J}\right]\right]\exp\left[-\frac{y}{\lambda_J}\right]}{\left[\left[H_s + \sqrt{H_s^2 - H^2}\right]^2 + H^2\exp\left[-\frac{2y}{\lambda_J}\right]\right]^2}, \tag{2.34}$$

$$f(y) = 1 - 4r(T)\frac{H^2\left[H_s + \sqrt{H_s^2 - H^2}\right]^2\exp\left[-\frac{2y}{\lambda_J}\right]}{\left[\left[H_s + \sqrt{H_s^2 - H^2}\right]^2 + H^2\exp\left[-\frac{2y}{\lambda_J}\right]\right]^2}. \tag{2.35}$$

The Meissner solution persists up to the field $H_s = (ep\lambda_J)^{-1}$ that should be regarded as the superheating field of the Meissner state. This fact was established for the LD model by Buzdin and Feinberg [5]. A self-consistent solution of the type (2.32)-(2.35) was first obtained in the framework of the microscopic theory [2]. In fields $H > H_s$, only vortex solutions are possible.

2. The lower critical field $H_{c1\infty}$. Vortex planes

Consider now an infinite (in the y direction) LD superconductor with $r(T) \ll 1$, $L_{y1} \to -\infty$, $L_{y2} \to +\infty$, and $j(y) \to 0$, $h(y) \to 0$ for $y \to \pm\infty$. We are interested in topological solutions of Eqs. (2.24), (2.26), (2.28), (2.29) for this situation. The requirement that the Josephson term be a minimum should now be understood as the condition that the density of the Josephson energy vanish at $y \to \pm\infty$. Thus, the appropriate boundary conditions are

$$\phi(-\infty) = 0, \quad \phi(+\infty) = \pm 2\pi, \quad \frac{d\phi}{dy}(\pm\infty) = 0, \quad f(\pm\infty) = 1. \tag{2.36}$$

[Note that aside from $\phi(+\infty)-\phi(-\infty)=\pm 2\pi$ no other topological boundary conditions are possible. This fact can be proved analogously to the well-known case of the sine-Gordon model [20].]

The desired solutions up to first order in the small parameter $r(T)$ are given by

$$\phi(y)=\pm 4\arctan\exp\left[\frac{y}{\lambda_J}\right], \tag{2.37}$$

$$h(y)=\pm H_s\cosh^{-1}\left[\frac{y}{\lambda_J}\right], \tag{2.38}$$

$$j(y)=\mp 2j_0\cosh^{-2}\left[\frac{y}{\lambda_J}\right]\sinh\left[\frac{y}{\lambda_J}\right], \tag{2.39}$$

$$f(y)=1-4r(T)\frac{\exp\left[-\frac{2|y|}{\lambda_J}\right]}{\left[1+\exp\left[-\frac{2|y|}{\lambda_J}\right]\right]^2}. \tag{2.40}$$

These solutions explicitly satisfy the usual conditions of the phase and flux quantization. Indeed, consider a closed rectangular contour Γ joining the points $\left(-\frac{N}{2}p,-\infty\right)$, $\left(-\frac{N}{2}p,+\infty\right)$, $\left(+\frac{N}{2}p,+\infty\right)$ and $\left(+\frac{N}{2}p,-\infty\right)$. The total change of the phase along this contour for the "plus" sign in (2.37) is

$$\Delta_\Gamma\phi=\int_{-\infty}^{+\infty}dy\frac{d\phi_{+\frac{N}{2}}(y)}{dy}+\int_{+\infty}^{-\infty}dy\frac{d\phi_{-\frac{N}{2}}(y)}{dy}=2\pi N.$$

Analogously, the total flux through this contour is

$$\Phi_\Gamma=Np\int_{-\infty}^{+\infty}dyh(y)=N\Phi_0,$$

where $\Phi_0=\pi/e$ is the flux quantum. Thus, the solution with the "plus" sign describes a chain of Josephson vortices positioned in the plane $y=0$ (one vortex per each barrier). Such a solution was first obtained in the framework of the microscopic theory [2] and termed "a vortex plane". The solution with the "minus" sign in (2.37) describes a chain of Josephson antivortices in the plane $y=0$ (i.e., "an antivortex plane").

By inserting (2.37) with the "plus" sign and (2.40) into (2.22) and comparing the result with the free energy of the Meissner state, we derive the lower critical field $H_{c1\infty}$, at which the vortex-plane solution becomes energetically favorable:

$$H_{c1\infty}=\frac{2}{\pi}H_s=\frac{2}{\pi}\frac{\Phi_0}{\pi p\lambda_J}. \tag{2.41}$$

Note that $h(0)=H_s>H_{c1\infty}$. This means that the penetration of Josephson vortices at fields $H_{c1\infty}<H<H_s$ can be prevented by a surface barrier, which should result in hysteretic behavior of magnetization [2]. Finally, we point out that simultaneous Josephson vortex penetration, envisaged by the vortex-plane solution, and hysteresis in the magnetization have recently been observed experimentally on artificial low-temperature superconducting superlattices Nb/Si [22].

3. The vortex state in intermediate fields

Now we turn to finite-size (in the y direction) LD superconductors with $r(T)\ll 1$, $-L_{y1}=L_{y2}\equiv W/2$, in the field range $H_s\ll H\ll H_{c2\infty}$ ($H_{c2\infty}$ is the upper critical field) and in the absence of externally applied current ($I=0$). The boundary conditions on ϕ have the form

$$\frac{1}{2ep}\frac{d\phi}{dy}\left(\pm\frac{W}{2}\right)=H. \tag{2.42}$$

Under these conditions, the phase difference up to first order in the small parameter H_s^2/H^2 is

$$\phi(y) = 2epHy + \pi N_v(H) - \frac{(-1)^{N_v}}{4}\frac{H_s^2}{H^2}\left[\sin\left(2epHy\right) - 2epHy\cos\left(epWH\right)\right]. \tag{2.43}$$

The constant of integration $\pi N_v(H)$ accounts for the requirement that the Josephson term in the free energy be a minimum. The "topological index" N_v corresponds to the number of vortex planes and is a singular function of the applied field H:

$$N_v(H) = \left[\frac{epWH}{\pi}\right] = \left[\frac{\Phi}{\Phi_0}\right]. \tag{2.44}$$

Here $[u]$ means the integer part of u, and $\Phi = pWH$ is the flux through one barrier.

By the use of (2.43), we derive the following expressions for the physical quantities up to first order in the small parameters $r(T)$ and H_s^2/H^2:

$$h(y) = H\left[1 - \frac{(-1)^{N_v}}{4}\frac{H_s^2}{H^2}\left[\cos\left(2epHy\right) - \cos\left(epWH\right)\right]\right], \tag{2.45}$$

$$j(y) = (-1)^{N_v} j_0 \sin\left(2epHy\right), \tag{2.46}$$

$$f(y) = 1 - \frac{r(T)}{2}\left[1 - \frac{(-1)^{N_v}\cos\left(2epHy\right)}{1+2\left[ep\zeta(T)H\right]^2} - \frac{\sqrt{2}ep\zeta(T)H\left|\sin\left(epWH\right)\right|}{1+2\left[ep\zeta(T)H\right]^2}\frac{\cosh\frac{\sqrt{2}y}{\zeta(T)}}{\sinh\frac{W}{\sqrt{2}\zeta(T)}}\right]. \tag{2.47}$$

In the limit $W \gg \zeta(T)$, $|y| \ll W/2$, equation (2.47) becomes

$$f(y) = 1 - \frac{r(T)}{2}\left[1 - \frac{(-1)^{N_v}\cos\left(2epHy\right)}{1+2\left[ep\zeta(T)H\right]^2}\right]. \tag{2.48}$$

The vortex solution (2.43), (2.45), (2.48) for $N_v = 2m$ (m is an integer) was first obtained by Theodorakis [8].

From Eq. (2.44) with $N_v(H) = 1$, we derive the lower critical field H_{c1W} in a finite along the layers superconductor with $W \ll \lambda_J$:

$$H_{c1W} = \frac{\pi}{epW} = \frac{\pi^2}{2}H_{c1\infty}\frac{\lambda_J}{W} \gg H_{c1\infty}. \tag{2.49}$$

For the magnetization $M = \frac{1}{4\pi W}\int\limits_{-\infty}^{+\infty} dyh(y) - \frac{H}{4\pi}$ we obtain:

$$M(H) = -\frac{H_s^2}{16\pi H}\left[\frac{\left|\sin\left(epWH\right)\right|}{epWH} - (-1)^{N_v}\cos\left(epWH\right)\right]. \tag{2.50}$$

The magnetization (2.50) shows distinctive oscillatory behavior and discontinuities when $epWH/\pi$ approaches an integer, i.e when a vortex plane penetrates or leaves the superconductor. For $H \gg \Phi_0/pW$, $\frac{N_v\Phi_0}{pW} < H < \left(N_v + \frac{1}{2}\right)\frac{\Phi_0}{pW}$, the LD superconductor exhibits a small paramagnetic effect, i.e. $M(H) > 0$. (Note that oscillations and jumps of magnetization due to Josephson vortex penetration have been experimentally observed on superconducting superlattices Nb/Si [22].)

4. Fraunhofer oscillations of the critical Josephson current

Consider the case of a finite-size (along the layers) LD superconductor with $r(T) \ll 1$, $-L_{y1} = L_{y2} \equiv W/2$ in the presence of an externally applied current I in the x direction. The boundary conditions on ϕ now are

$$\frac{1}{2ep}\frac{d\phi}{dy}\left(\pm\frac{W}{2}\right) = H \pm 2\pi I. \tag{2.51}$$

Assuming $W \ll \lambda_J$, we obtain the solution up to first order in the small parameters $r(T)$ and W^2/λ_J^2:

$$\phi(y) = 2epHy + \pi N_v(H) + \varphi$$

$$- \frac{(-1)^{N_v}}{4}\frac{W^2}{\lambda_J^2}(epWH)^{-2}\left[\sin(2epHy+\varphi) - 2epHy\cos(epWH)\cos\varphi - \sin\varphi\right], \tag{2.52}$$

$$I(\varphi, H) = \int_{-\frac{W}{2}}^{+\frac{W}{2}} dy j(y) = \frac{j_0}{epH}\left|\sin(epWH)\right|\sin\varphi, \tag{2.53}$$

$$h(y) = H\left[1 - \frac{(-1)^{N_v}}{4}\frac{W^2}{\lambda_J^2}(epWH)^{-2}\left[\cos(2epHy+\varphi) - \cos(epWH)\cos\varphi\right]\right], \tag{2.54}$$

$$f(y) = 1 - r(T)\left[1 - \frac{(-1)^{N_v}\cos(2epHy+\varphi)}{1+2\left[ep\zeta(T)H\right]^2}\right.$$

$$- \frac{\sqrt{2}ep\zeta(T)H\left|\sin(epWH)\right|\cos\varphi}{1+2\left[ep\zeta(T)H\right]^2}\frac{\cosh\frac{\sqrt{2}y}{\zeta(T)}}{\sinh\frac{W}{\sqrt{2}\zeta(T)}}$$

$$\left. - \frac{(-1)^{N_v}\sqrt{2}ep\zeta(T)H\cos(epWH)\sin\varphi}{1+2\left[ep\zeta(T)H\right]^2}\frac{\sinh\frac{\sqrt{2}y}{\zeta(T)}}{\cosh\frac{W}{\sqrt{2}\zeta(T)}}\right]. \tag{2.55}$$

The phase shift $\pi N_v(H)$, induced by N_v vortex planes, assures the condition of a minimum of the Josephson energy. The field-independent phase shift φ ($|\varphi| \leq \pi/2$) parameterizes the total Josephson current I given by (2.53). Equation (2.53) yields the well-known Fraunhofer pattern in layered superconductors [6,2]. Note that the first zero of the Fraunhofer pattern, by (2.49), corresponds to the lower critical field H_{c1W}. (See Ref. [2] for the explanation of the Fraunhofer pattern in terms of the pinning of the vortex planes by the edges of the superconductor.) In the absence of the transport current, i. e., for $\varphi = 0$, equations (2.52), (2.54), (2.55) reduce, respectively, to (2.43), (2.45) and (2.47).

5. The upper critical field $H_{c2\infty}(T)$

Here we consider an infinite (in the y direction) LD superconductor with $-L_{y1} = L_{y2} \equiv W/2 \to +\infty$, subject to boundary conditions on the phase of the type (2.42). Supposing that at the upper critical field $H = H_{c2\infty}$ the transition to the normal phase is of the second-order type, f^2 can be considered as a small parameter, and equations (2.24), (2.26) become:

$$f(y) + \zeta^2(T)\frac{d^2 f(y)}{dy^2} - r(T)\left[1 - \cos\phi(y)\right]f(y) = 0, \tag{2.56}$$

$$\frac{d^2\phi(y)}{dy^2} = 0. \tag{2.57}$$

The relevant solution of Eq. (2.57) is

$$\phi(y) = 2epHy + \pi N_v(H). \tag{2.58}$$

[Compare with (2.43).]. The substitution of (2.58) into (2.56) yields

$$\frac{d^2 f(t)}{dt^2} + \left[A(T,H) - (-1)^{N_v+1} q(H)\cos 2t\right] f(t) = 0, \tag{2.59}$$

$$A(T,H) \equiv \frac{[1-r(T)]}{[ep\zeta(T)H]^2}, \quad q(H) \equiv \frac{r(T)}{2\,[ep\zeta(T)H]^2} = \frac{\alpha_{ph}}{[ep\xi_{ph}H]^2},$$

where we have introduced a dimensionless variable $t \equiv epHy$ and the notation $\zeta(0) \equiv \xi_{ph}$. Hence one gets two independent equations: for the odd $N_v = 2m+1$ ($m = 0, 1, 2, \ldots$) and the even $N_v = 2m$ number of vortex planes. Both of them have the usual form of the Mathieu equations [21]. [Note that for $N_v = 2m$ Eq. (2.59) is well-known [3].]

The upper critical field H_{c2} is now determined by the smallest eigenvalue of (2.59):

$$A(T, H_{c2\infty}) = a_0(q_c), \tag{2.60}$$

where $q_c \equiv q(H_{c2\infty})$, and $a_0(q)$ $[a_0(-q) = a_0(q)]$ is the smallest eigenvalue of the Mathieu equation corresponding to the eigenfunctions $f_{N_v=2m+1}(t) \propto \mathrm{ce}_0(t,q)$ and $f_{N_v=2m}(t) \propto \mathrm{ce}_0(\pi/2 - t, q)$. [Note that the function $\mathrm{ce}_0(t,q)$ is strictly positive and periodic with the period π.] Explicitly, equation (2.60) reads:

$$\frac{\tau - 2\alpha_{ph}}{[ep\xi_{ph}H_{c2\infty}]^2} = a_0\left(\frac{\alpha_{ph}}{[ep\xi_{ph}H_{c2\infty}]^2}\right). \tag{2.61}$$

Equation (2.61) exhibits the well-known 3D-2D crossover of $H_{c2\infty}(T)$ [3], with the crossover temperature determined by $\tau^* = 2\alpha_{ph}$. As usual, it is of interest to consider two opposite limiting cases.

High temperatures, weak fields: $\tau \ll 2\alpha_{ph}, \quad H_{c2\infty} \ll \sqrt{\alpha_{ph}}/ep\xi_{ph}$.

In this 3D regime,

$$H_{c2\infty}(T) = \frac{1}{2\sqrt{\alpha_{ph}}}\frac{\tau}{ep\xi_{ph}} = \frac{1}{2\sqrt{\alpha_{ph}}}\frac{1}{ep\xi_{ph}}\left(1 - \frac{T}{T_{c0}}\right). \tag{2.62}$$

The superconductivity of the S-layers is strongly depressed by the vortex planes, which can be seen by comparing local maxima $f_{\max}$ with local minima $f_{\min}$ of the order parameter:

$$\frac{f_{\min}}{f_{\max}} = 2\sqrt{2}\exp\left[-2r(T)\right] \ll 1.$$

Low temperatures, strong fields: $2\left(1 - \frac{\tau}{2\alpha_{ph}}\right) \ll 1, \quad H_{c2\infty} \gg \sqrt{\alpha_{ph}}/ep\xi_{ph}$.

In this regime,

$$H_{c2\infty}(T) = \frac{\sqrt{\alpha_{ph}}}{2ep\xi_{ph}}\left(1 - \frac{\tau}{2\alpha_{ph}}\right)^{-\frac{1}{2}} = \frac{\alpha_{ph}}{\sqrt{2}ep\xi_{ph}}\frac{\sqrt{T_{c0}}}{\sqrt{T - T_{c0}(1 - 2\alpha_{ph})}}. \tag{2.63}$$

This expression diverges for $\tau \to \tau^* - 0$. The origin of this well-known unphysical divergence is the unrealistic assumption of the LD model of a negligible S-layer thickness. [In the microscopic theory [2], $H_{c2\infty}(T)$ is finite at any temperatures.] The spatial dependence of the order parameter is given by

$$f(y) \propto 1 - \frac{(-1)^{N_v} r(T)}{4\,[ep\zeta(T)H_{c2\infty}]^2}\cos\left(2epH_{c2\infty}y\right).$$

This spatial dependence is exactly the same as in the case of intermediate fields (2.48).

III. THE MEISSNER STATE IN THE FINITE LD MODEL

Let the LD superconductor occupy the region $[L_{x1} = 0 \le x \le L_{x2}] \times [L_{y1} = 0 \le y \le L_{y2}]$. The external magnetic field $\mathbf{H}$ ($0 \le H \le H_s$) is again applied along the z axis. The homogeneity along this axis is assumed ($W_z \to +\infty$). The Meissner state realizes under the conditions $L_{x2} \gg \lambda$, $L_{y2} \gg \lambda_J$, thus it is sufficient to consider the limiting case $L_{x2} \to +\infty$, $L_{y2} \to +\infty$.

This situation is described by the functional (2.2) with a minor change: the summation is now done over $n = 0, 1, 2, \ldots$ We assume that $r(T) \ll 1$. Boundary conditions of the type (2.9) are supposed to hold at $y = 0$, and $h(x, 0) = H$. The presence of an outer boundary at $x = 0$ is accounted for by the obvious boundary condition

$$H - h(+0, y) = \frac{p f_0^2(y)}{2e\lambda^2(T)} \left[\frac{d\phi_0(y)}{dy} - 2eA_y(0, y) \right]. \tag{3.1}$$

[Compare with (2.8).] The imposition of the boundary condition (3.1) implies a restriction on variations of $A_y(x, y)$: they must now satisfy the condition

$$\delta A_y(0, y) = 0. \tag{3.2}$$

The influence of the boundary at $y = 0$ must vanish for $y \to +\infty$, hence boundary conditions

$$\Phi_{n+1,n}(+\infty) = 0, \quad \frac{d\Phi_{n+1,n}}{dy}(+\infty) = 0. \tag{3.3}$$

For $x \to +\infty$, we must arrive at the solution of the infinite LD model (2.38)-(2.40), thus

$$\frac{d\phi_n(y)}{dy} - 2eA_y(np, y) \to 0, \quad n \to +\infty. \tag{3.4}$$

The minimization with respect to f_n leads to (2.16) and (2.17). Varying with respect to A_x under the condition $\delta A_x(x, 0) = \delta A_x(x, +\infty) = 0$ yields the Maxwell equations (2.6) in the regions $np < x < (n+1)p$ $(n = 0, 1, 2, \ldots)$. Taking variations with respect to A_y under the condition (3.2), we obtain the Maxwell equations (2.7) in the regions $np < x < (n+1)p$ $(n = 0, 1, 2, \ldots)$ and boundary conditions at the S-layers (2.8) for $n = 1, 2, \ldots$

The general solution of the Maxwell equations, subject to the above formulated boundary conditions, in the gauge (2.20), has the form

$$A(0, y) = \frac{r(T)}{4e\zeta^2(T)} \frac{1}{f_0^2(y)} \int_0^y du f_1(u) f_0(u) \sin\phi_{1,0}(u) + \frac{1}{2e} \frac{d\phi_0(y)}{dy},$$

$$A(x, y) = \left[4\pi j_0 \int_0^y du f_{n+1}(u) f_n(u) \sin\phi_{n+1,n}(u) + H \right] [x - (n+1)\, p] + \frac{1}{2e} \frac{d\phi_{n+1}(y)}{dy}$$

$$- \frac{r(T)}{4e\zeta^2(T)} \frac{1}{f_{n+1}^2(y)} \int_0^y du f_{n+1}(u) \left[f_n(u) \sin\phi_{n+1,n}(u) - f_{n+2}(u) \sin\phi_{n+2,n+1}(u) \right], \tag{3.5}$$

$$np < x \le (n+1)\, p, \quad n = 0, 1, 2, \ldots,$$

where the phase differences $\phi_{n+1,n}$ obey the solvability conditions

$$\frac{d\phi_{1,0}(y)}{dy} = 8\pi e j_0 p \int_0^y du f_1(u) f_0(u) \sin\phi_{1,0}(u) + 2epH$$

$$+ \frac{r(T)}{2\zeta^2(T)} \left[\frac{1}{f_1^2(y)} \int_0^y du f_1(u) \left[f_0(u) \sin\phi_{1,0}(u) - f_2(u) \sin\phi_{2,1}(u) \right] \right.$$

$$\left. + \frac{1}{f_0^2(y)} \int_0^y du f_1(u) f_0(u) \sin\phi_{1,0}(u) \right],$$

$$\frac{d\phi_{n+1,n}(y)}{dy} = 8\pi e j_0 p \int_0^y du f_{n+1}(u) f_n(u) \sin\phi_{n+1,n}(u) + 2epH$$

$$+\frac{r(T)}{2\zeta^2(T)}\left[\frac{1}{f_{n+1}^2(y)}\int_0^y du f_{n+1}(u)\left[f_n(u)\sin\phi_{n+1,n}(u) - f_{n+2}(u)\sin\phi_{n+2,n+1}(u)\right]\right.$$

$$\left.-\frac{1}{f_n^2(y)}\int_0^y du f_n(u)\left[f_{n-1}(u)\sin\phi_{n,n-1}(u) - f_{n+1}(u)\sin\phi_{n+1,n}(u)\right]\right], \quad n = 1, 2, \ldots \tag{3.6}$$

Equations (3.6) assure the continuity of the solution (3.5) at $x = np$ $(n = 0, 1, 2, \ldots)$. [Compare with Eqs. (A2), (A3) of the infinite LD model.] The obtained solution explicitly satisfies the current-conservation law

$$\frac{p}{8\pi e\lambda^2}\sum_{n=0}^{+\infty} f_n^2(y)\left[\frac{d\phi_n(y)}{dy} - 2eA_y(np, y)\right] + \int_0^y du\, j(u) = 0, \tag{3.7}$$

where $j(y) \equiv \lim_{n\to+\infty} j_{n+1,n}(y)$ is the density of the Josephson current given by (2.34). [Compare with the current-conservation law (2.12) of the infinite LD model.]

Note that, in contrast to the infinite LD model, the minimization with respect to the phases ϕ_n now is not possible. Indeed, a variation of the phase at the mth layer, $\delta\phi_m$, would induce, by (3.7), a non-vanishing variation of the vector potential at $x = 0$:

$$\delta A_y(0, y) = \frac{1}{2e}\frac{f_m^2(y)}{f_0^2(y)}\frac{d\delta\phi_m(y)}{dy}.$$

Such a variation is not allowed by the condition (3.2). This observation explains the role of the arbitrariness contained in the system of the Maxwell equations of the infinite LD model, discussed in the previous section. In the infinite case, the variations $\delta A_y(\pm\infty, y)$ were arbitrary, which allowed the system to adjust boundary conditions at $x \to \pm\infty$, so as to minimize the free energy with respect to the phases.

We are interested in the behavior of (3.5) in the asymptotic region $y \to +\infty$. The second set of the conditions (3.3), applied to (3.6), in first order in the small parameter $r(T)$ yields

$$I_{2,1} - \left(2 + \frac{p^2}{\lambda^2}\right) I_{1,0} - \frac{p^2 H}{4\pi\lambda^2} = 0,$$

$$I_{n+2,n+1} - \left(2 + \frac{p^2}{\lambda^2}\right) I_{n+1,n} + I_{n,n-1} - \frac{p^2 H}{4\pi\lambda^2} = 0, \quad n = 1, 2, \ldots, \tag{3.8}$$

where

$$I_{n+1,n} \equiv j_0 \int_0^{+\infty} du f_{n+1}(u) f_n(u) \sin\phi_{n+1,n}(u), \quad n = 0, 1, 2, \ldots$$

is the total Josephson current between the $(n+1)$th and the nth layers. The local magnetic field inside the barriers $np < x < (n+1)\,p$ $(n = 0, 1, 2, \ldots)$ in the asymptotic region $y \to +\infty$ is given by

$$h_{n+1} \equiv h(x, +\infty) = 4\pi I_{n+1,n} + H. \tag{3.9}$$

With the help of the quantities h_n, equations (3.8) can be rewritten in the form of a recursion relation

$$h_{n+2} - \left(2 + \frac{p^2}{\lambda^2}\right) h_{n+1} + h_n = 0, \quad n = 0, 1, 2, \ldots, \tag{3.10}$$

subject to boundary conditions

$$h_0 = H, \quad h_n \to 0, \quad n \to +\infty. \tag{3.11}$$

The solution of (3.10), (3.11) is straightforward:

$$h_n = H\left[1 + \frac{p^2}{2\lambda^2} - \frac{p}{\lambda}\sqrt{1 + \frac{p^2}{4\lambda^2}}\right]^n. \tag{3.12}$$

Assuming $p \ll \lambda$, we get

$$h(x, +\infty) = H \exp\left[-\frac{(n+1)\,p}{\lambda(T)}\right], \quad np < x < (n+1)\,p, \quad n = 0, 1, 2, \ldots \tag{3.13}$$

The intralayer currents are given by

$$J_n(+\infty) = \frac{1}{4\pi p}\,(h_n - h_{n+1}) = \frac{H}{4\pi\lambda(T)} \exp\left[-\frac{np}{\lambda(T)}\right], \quad n = 0, 1, 2, \ldots \tag{3.14}$$

The order parameters are

$$f_n(+\infty) = 1 - 2e^2\zeta^2(T)\lambda^2(T)H^2 \exp\left[-\frac{2np}{\lambda(T)}\right], \quad n = 0, 1, 2, \ldots \tag{3.15}$$

Equations (3.13)- (3.15) describe the Meissner state in the region $[0 \le x < +\infty) \times (\lambda_J \ll y < +\infty)$. In the region $(\lambda \ll x < +\infty) \times [0 \le y < +\infty)$, the solution is given by (2.38)-(2.40). As in the case of the infinite LD model, the upper bound of the existence of these solutions is $H = H_s$. Unfortunately, in the region $[0 \le x < \lambda) \times [0 \le y < \lambda_J)$, an analytical solution to Eqs. (2.16) and (3.6) is not possible.

Equations of the type (2.16), (3.5) and (3.6), subject to topological boundary conditions on $\phi_{n+1.n}$, in principle, describe Josephson vortex configurations. In contrast to the equations of the infinite LD model, these equations do not preclude inhomogeneous in the layering direction vortex solutions.

IV. DISCUSSION

We have solved the problem of exact minimization of the LD functional in both the infinite and the finite cases. We have shown that the LD model belongs to a class of singular field theories [17,18]: the Maxwell equations of this model contain constraints [the current conservation laws (2.12), (3.7)] on the phases and the vector potential at different superconducting layers. Such constraints, resulting from gauge invariance combined with inherent discreteness, are typical of layered superconductors. [See Eq. (26) of the microscopic theory [2].] Unfortunately, the current conservation laws (2.12), (3.7) were completely overlooked in previous publications on the LD model.

By taking into account the current-conservation law (2.12), we have minimized the free energy of the infinite LD model with respect to the phases, obtaining a closed, complete, self-consistent system of mean-field equations (2.23)-(2.29). We show that relations of the type (2.11), erroneously regarded as "equations minimizing the free energy with respect to the phases" [15], are, in reality, mere consequences of the Maxwell equations (2.6)-(2.8). By considering non-vanishing first-order variations of (2.2) caused by the variation (A5) of an inhomogeneous solution to the Maxwell equations, we prove that the infinite LD model does not admit solutions in the form of isolated Josephson vortices.

The exact mean-field equations (2.23)-(2.29) contain the whole physics of the infinite LD model in parallel magnetic fields. In particular, they reproduce such well-known limiting cases as the Meissner state [5], Fraunhofer oscillations of the critical Josephson current [6], the upper critical field $H_{c2}(T)$ [7,3], and the vortex state [8] in the intermediate field regime. All previous calculations of these effects were based on a physical assumption of the homogeneity of the solution in the layering direction. We provide a rigorous mathematical justification of this assumption by proving that the homogeneity of the solution is one of the necessary and sufficient conditions of a minimum of (2.2). Moreover, our approach allows us to make self-consistent refinements on these results by obtaining exact analytical expressions for all physical quantities of interest up to leading order in small parameters, which substantially elucidates the physics.

We have obtained an exact topological solution (2.37)-(2.40) to Eqs. (2.23)-(2.29), describing a chain of Josephson vortices (a vortex plane). This solution clearly demonstrates that, contrary to previous suggestions [9,10], Josephson vortices of the infinite LD model form simultaneously and coherently (one vortex per each barrier) at the lower critical field $H_{c1\infty}$, given by (2.41). Successive penetration of the vortex planes at higher fields is accompanied by oscillations

and jumps of the magnetization, as described by (2.50). We show that the vortex-plane solutions of the infinite LD model persist up to the upper critical field $H_{c2\infty}(T)$ in the whole temperature range.

Our consideration of the finite LD model illuminates the role of the boundary effect. Thus, the imposition of the boundary condition (3.1) completely eliminates all unphysical degrees of freedom and makes minimization with respect to the phases impossible. The explicit solution (3.5), (3.6) to the Maxwell equations, in contrast to the infinite case, does not preclude the existence of localized Josephson vortex configurations. Making use of this solution, we obtain the first, to our mind, self-consistent description of the Meissner state of the finite LD model [Eqs. (3.13)-(3.15) and (2.38)- (2.40)].

All the above results stand in full agreement with our previous consideration of layered superconductors [2] based on a completely different, microscopic approach. Our discussion (Appendix B) of relationship to the microscopic theory [2] clarifies a microscopic background of the phenomenological parameters of the LD model and casts light on its actual domain of validity.

As regards the experimental status of the problem, simultaneous Josephson vortex penetration into all the barriers, as described in our paper, has been recently observed on artificial low-T_c superlattices Nb/Si [22]. Oscillations and jumps of the magnetization, accompanying this penetration, have also been observed [22]. Concerning the reported observation of localized Josephson vortex configurations in layered high-T_c superconductors [23], the boundary effect discussed in section III of our paper may account for this situation. Moreover, the presence of irregularities within the layered structure (e.g., stacking faults) can substantially modify the physical picture: such irregularities should serve as pinning centers for isolated Josephson vortices. We hope that our exact results will stimulate further theoretical and experimental investigation in these directions.

APPENDIX A: THE EXPLICIT SOLUTION TO THE MAXWELL EQUATIONS OF THE INFINITE LD MODEL

In the gauge $A_x = 0$, the explicit solution of Eqs. (2.6)-(2.8) on the intervals

$$(n-1)p < x \leq np, \quad n = 0, \pm 1, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{A1}$$

subject to boundary conditions (2.9) at $y = L_{y1}$ and $h(L_{y1}) = H - 2\pi I$, has the form

$$A_y(x,y) = \left[4\pi j_0 \int_{L_{y1}}^{y} du f_n(u) f_{n-1}(u) \sin\phi_{n,n-1}(u) + H - 2\pi I \right] (x - np) + \frac{1}{2e}\frac{d\phi_n(y)}{dy}$$

$$- \frac{r(T)}{4e\zeta^2(T)} \frac{1}{f_n^2(y)} \int_{L_{y1}}^{y} du f_n(u) \left[f_{n-1}(u) \sin\phi_{n,n-1}(u) - f_{n+1}(u) \sin\phi_{n+1,n}(u) \right], \tag{A2}$$

where the phase differences $\phi_{n,n-1} = \phi_n - \phi_{n-1}$ obey the solvability conditions

$$\frac{d\phi_{n+1,n}(y)}{dy} = 8\pi e j_0 p \int_{L_{y1}}^{y} du f_{n+1}(u) f_n(u) \sin\phi_{n+1,n}(u) + 2ep\,(H - 2\pi I)$$

$$+ \frac{r(T)}{2\zeta^2(T)} \left[\frac{1}{f_{n+1}^2(y)} \int_{L_{y1}}^{y} du f_{n+1}(u) \left[f_n(u) \sin\phi_{n+1,n}(u) - f_{n+2}(u) \sin\phi_{n+2,n+1}(u) \right] \right.$$

$$\left. - \frac{1}{f_n^2(y)} \int_{L_{y1}}^{y} du f_n(u) \left[f_{n-1}(u) \sin\phi_{n,n-1}(u) - f_{n+1}(u) \sin\phi_{n+1,n}(u) \right] \right]. \tag{A3}$$

This infinite system of integrodifferential equations assures the continuity of the solution (A2) at $x = np$ ($n = 0, \pm 1, \pm 2, \ldots$). For $f_n = 1$, equations (A3) reduce to an infinite set of second-order non-linear differential equations [13]. Unfortunately, an explicit solution for the vector potential of the type (A2) was not found in previous publications.

Consider the variation of the solution (A2) induced by a variation of the phase at one of the S-layers, $\delta\phi_n(y)$. As there is only one constraint on $\frac{d\phi_n}{dy}$ and the y components of the vector potential at $x = np$, namely the current-conservation law (2.12), such a variation affects the solution (A2) only on one of the intervals (A1), say, $(m-1) < x \leq mp$. Making use of (2.13), we rewrite the solution (A2) on this interval as

$$A_y(x,y) = \left[4\pi j_0 \int\limits_{L_{y1}}^{y} du f_m(u) f_{m-1}(u) \sin\phi_{m,m-1}(u) + H - 2\pi I\right](x - mp)$$

$$+\frac{1}{2e}\frac{d\phi_m(y)}{dy} + \frac{1}{f_m^2(y)}\sum_{n\neq m} f_n^2(y)\left[\frac{1}{2e}\frac{d\phi_n(y)}{dy} - A_y(np,y)\right]. \tag{A4}$$

In equation (A4), all $\frac{d\phi_n}{dy}$ should be considered as independent. Thus, the desired variation is

$$\delta A_y(x,y) = \frac{1}{2e}\frac{f_n^2(y)}{f_m^2(y)}\frac{d\delta\phi_n(y)}{dy}, \tag{A5}$$

where $n = 0, \pm 1, \pm 2, \ldots$

APPENDIX B: RELATIONSHIP TO THE MICROSCOPIC THEORY

The free energy functional of the microscopic theory [2], after the minimization with respect to $\mathbf{A}$, has the form

$$\Omega\left[f,\phi;H\right] = \frac{H_c^2(T)}{4\pi} W_x W_z \int\limits_{L_{y1}}^{L_{y2}} dy \left[\frac{a}{p}\left[-f^2(y) + \frac{1}{2}f^4(y) + \zeta^2(T)\left[\frac{df(y)}{dy}\right]^2\right.\right.$$

$$\left. + \frac{\zeta^2(T)}{12}\left(\frac{a}{p}\right)^2\left[\frac{d\phi(y)}{dy}\right]^2 f^2(y) + \frac{\alpha\zeta^2(T)}{a\xi_0}\left[1-\cos\phi(y)\right]f^2(y)\right]$$

$$\left. + 4e^2\zeta^2(T)\lambda^2(T)\left[\frac{1}{2ep}\frac{d\phi(y)}{dy} - H\right]^2\right], \tag{B1}$$

where ξ_0 is the BCS coherence length, $\zeta(T)$ and $\lambda(T)$ are the GL coherence length and the penetration depth, respectively, a is the S-layer thickness,

$$\alpha = \frac{3\pi^2}{7\zeta(3)}\int\limits_0^1 dt t D(t) \ll 1, \tag{B2}$$

with $D(t)$ being the tunneling probability of the barrier between two successive S-layers. The rest of notation is the same as in (2.23). Expression (B1) applies to the temperature range (2.1) and the S-layer thicknesses meeting the condition $\xi_0 \ll a \ll \zeta(T), \lambda(T)$.

Consider the LD limit of (B1), when $a \ll p$. In this limit, the average kinetic energy of the intralayer currents, i.e. the term proportional to a^3/p^3, should be dropped. However, the microscopic functional (B1) does not reduce to the corresponding LD functional (2.22) because of the presence of the first order factor a/p. (In the LD model this factor is unrealistically taken to be unity.) Nevertheless, as can be easily seen by minimizing (B1) with respect to f and ϕ, the microscopic mean-field equations in this limit formally coincide with the LD equations (2.23)-(2.29), if one

identifies $r(T)$ with the microscopic parameter $\frac{\alpha\zeta^2(T)}{a\xi_0}$ and the LD quantity j_0 with the microscopic expression for the critical Josephson current of a single junction

$$j_0 = \frac{7\zeta(3)\alpha}{6} eN(0)\xi_0\Delta^2(T),$$

where $N(0)$ is the one-spin density of states at the Fermi level.

The role of the first-order factor a/p becomes evident when one considers the penetration of an external parallel magnetic field in the layering direction. As can be shown on the basis of the microscopic equations [2], the exponential falloff of the magnetic field occurs on the length scale of the effective penetration depth $\lambda_{eff} = \lambda\sqrt{\frac{p}{a}}$, whereas the LD model gives $\lambda_{eff} = \lambda$. [See Eq. (3.13).]

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